THE QUADRATIC APPROXIMATION LEMMA AND DECOMPOSITIONS OF SUPERLATIVE INDEXES

by

W. Erwin Diewert

October 2000

Discussion Paper No.: 00-15

DEPARTMENT OF ECONOMICS
THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, CANADA V6T 1Z1

http://web.arts.ubc.ca/econ

THE QUADRATIC APPROXIMATION LEMMA AND DECOMPOSITIONS OF SUPERLATIVE INDEXES

W. Erwin Diewert,[1]
Department of Economics,
University of British Columbia,
Vancouver, B. C.,
Canada, V6T 1Z1.
Email: diewert@econ.ubc.ca

October 30,2000 version.

Abstract

It was shown in 1976 that a difference in a quadratic function of N variables evaluated at two points is exactly equal to the sum of the arithmetic average of the first order partial derivatives of the function evaluated at the two points times the differences in the independent variables. In the present paper, this result is generalized and the resulting generalized quadratic approximation lemma is used to establish all of the superlative index number formulae that were derived in Diewert (1976). In addition, some new exact decompositions of the percentage change in the Fisher and Walsh superlative indexes into N components are derived. Each component in this decomposition represents the contribution of a change in a single independent variable to the overall percentage change in the index. Finally, these components are given economic interpretations.

Keywords

Superlative index, quadratic approximations, percentage change decompositions, price indexes, quantity indexes, Fisher ideal index.

Journal of Economic Literature Classification Numbers

C43, E31,O47

[1] University of British Columbia and NBER. The author thanks the Social Sciences and Humanities Research Council of Canada for financial support and Bert Balk and Rolf Färe for helpful comments.

THE QUADRATIC APPROXIMATION LEMMA AND DECOMPOSITIONS OF SUPERLATIVE INDEXES

W. Erwin Diewert,[1]
Department of Economics,
University of British Columbia,
Vancouver, B. C.,
Canada, V6T 1Z1.
Email: diewert@econ.ubc.ca

October 30, 2000 version.

1. Introduction

Let $F(z_1,\ldots,z_N)$ be a quadratic function of N variables, $(z_1,\ldots,z_N) \equiv z$; i.e., define F as follows:

$$(1)\quad F(z) \equiv a_0 + \sum_{n=1}^{N} a_n z_n + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\, z_i z_j$$

where $a_{ij} = a_{ji}$ for all i and j. It is well known that a second order Taylor series approximation to a quadratic function will *exactly* reproduce the quadratic function. It is not so well known that the arithmetic average of two linear approximations will also reproduce a quadratic function *exactly*. To see this, write the linear approximation to F(z) around the point $z^0 \equiv (z_1^{\,0},\ldots,z_N^{\,0})$ as

$$(2)\quad F(z) \approx F(z^0) + \quad F(z^0)\cdot [z - z^0] \equiv F(z^0) + \sum_{n=1}^{N} F_n(z^0)[z_n - z_n^{\,0}]$$

where $\quad F(z^0) \equiv [\partial F(z^0)/\partial z_1,\ldots,\partial F(z^0)/\partial z_N] \equiv [F_1(z^0),\ldots,F_N(z^0)]$ is the vector of first order partial derivatives of F evaluated at the point z^0 and $x\cdot y \equiv \sum_{n=1}^{N} x_n y_n$ denotes the inner product of the vectors x and y. The linear approximation to F around another point z^1 is

$$(3)\quad F(z) \approx F(z^1) + \quad F(z^1)\cdot [z - z^1] \equiv F(z^1) + \sum_{n=1}^{N} F_n(z^1)[z_n - z_n^{\,1}].$$

Now let $z = z^1$ in (2) and $z = z^0$ in (3) and treat the two approximations as equalities. Taking the arithmetic average of the resulting two equations and rearranging terms yields the following equation:

$$\begin{aligned}(4)\quad F(z^1) - F(z^0) &= (1/2)[\quad F(z^0) + \quad F(z^1)]\cdot [z^1 - z^0] \\ &= \sum_{n=1}^{N} (1/2)[F_n(z^0) + F_n(z^1)][z_n^{\,1} - z_n^{\,0}].\end{aligned}$$

It can be verified by substituting the F defined by (1) into (4) that if F is quadratic, then (4) holds *exactly* for any two points, z^0 and z^1.[2] Note that this result shows that taking an

---

[1] University of British Columbia and NBER. The author thanks the Social Sciences and Humanities Research Council of Canada for financial support and Bert Balk and Rolf Färe for helpful comments.

[2] Diewert (1976; 118) and Lau (1979) showed that the converse result also holds; i.e., if (4) holds, then F must be quadratic. Diewert called this result the quadratic approximation lemma.

average of two first order approximations yields the equivalent of a second order approximation.

Diewert (1976; 118-121) used the fact that (4) holds as an identity for quadratic functions to prove that the Törnqvist (1936) (1937) quantity index is exact for a translog aggregator function and that the Törnqvist price index is exact for a translog unit cost function.[4] Diewert established these results by taking a simple transformation of (4).

It turns out that other transformations of the *quadratic identity* (4) can be used to establish the exactness of all of the major families of superlative index numbers.[5] We show this in section 4 below.

In section 2, we provide the economic framework for our index number results.

In section 3, we provide a rather general transformation of the quadratic identity (4). We then show how a special case of this general result yields the translog results.

In section 4, we specialize our general transformation of (4) to yield the exactness of the quadratic mean of order r indexes. Thus our transformation of (4) provides a unified framework for deriving the commonly used superlative index number formulae.

In section 5, we specialize the results of section 4 to the case where r equals 2. This specialization allows us to obtain additive percentage change decompositions for the Fisher (1922) ideal price and quantity indexes.

In section 6, we specialize the results of section 4 to the case where r equals 1. This specialization allows us to obtain additive percentage change decompositions for some indexes that were originally defined by Walsh (1901) (1921).

In section 5, we find that our additive percentage change decompositions for the Fisher ideal price and quantity indexes are not unique. Thus it is important to provide some sort of an axiomatic or economic justification for any particular additive percentage change decomposition. In section 7, we provide economic interpretations for our preferred decompositions.

Section 8 concludes. We conclude that the decompositions that we obtain for the Törnqvist[6], Fisher and Walsh indexes are particularly attractive.

2. The Economic Framework

For simplicity, we consider a consumer[7] who minimizes the cost of achieving a given utility level in two periods where the *utility or aggregator function* f(q) is (positively)

---

[4] The translog functional form was introduced into the economics literature by Christensen, Jorgenson and Lau (1971) (1973).

[5] A superlative index number formula is exact for a flexible functional form; see Diewert (1976).

[6] The Törnqvist decomposition was obtained earlier by Diewert and Morrison (1986) and Kohli (1990).

linearly homogeneous, positive for positive q, nondecreasing and concave function in the N variables, $q \equiv (q_1,\ldots,q_N)$. We assume that we can observe the price vectors that the consumer faces during these two periods, say $p^t \equiv (p_1^t,\ldots,p_N^t)$ for t = 0,1, and the quantity vectors chosen for the two periods, say $q^t \equiv (q_1^t,\ldots,q_N^t)$ for t = 0,1. The *unit cost function* c that corresponds to the aggregator function f is defined as the minimum cost of achieving the utility level 1; i.e., for each vector of positive commodity prices p, define c by:[8]

(5) $c(p) \equiv \min_q \{p\cdot q : f(q) = 1\}$.

Under our assumptions on consumer behavior, the observed period t expenditure on the N commodities, $p^t\cdot q^t = \sum_{n=1}^N p_n^t q_n^t$, will equal the product of the period t utility level, $f(q^t)$, times the period t unit cost, $c(p^t)$:[9]

(6) $p^t\cdot q^t = c(p^t)f(q^t)$; t = 0,1.

Thus taking ratios of the period 1 expenditures on the N commodities to the period 0 observed expenditures, we obtain:

(7) $$\begin{aligned} p^1\cdot q^1 / p^0\cdot q^0 &= [c(p^1)f(q^1)]/[c(p^0)f(q^0)] \\ &= [c(p^1)/c(p^0)][f(q^1)/f(q^0)]. \end{aligned}$$

The term $[c(p^1)/c(p^0)]$ on the right hand side of (7) can be interpreted as the consumer's "true" price index[10] and the term $[f(q^1)/f(q^0)]$ can be interpreted as the consumer's "true" quantity index.

If the unit cost function c(p) is differentiable with respect to the components of the price vector p, then Shephard's (1953; 11) Lemma implies the following useful equations:[11]

(8) $c(p^t)/c(p^t) = q^t /p^t\cdot q^t$; t = 0,1.

If the aggregator function f(q) is differentiable with respect to the components of q, the Wold's (1944; 69-71) (1953; 145) Identity implies the following equally useful equations:[12]

(9) $f(q^t)/f(q^t) = p^t /p^t\cdot q^t$; t = 0,1.

---

[7] Alternatively, the same theory applies to a producer who minimizes the cost of achieving a given level of output in two periods, where $f(q_1,\ldots,q_N)$ is the maximum output that can be produced by the vector of inputs $q \equiv (q_1,\ldots,q_N)$. This is the framework used by Shephard (1953), Samuelson and Swamy (1974) and Diewert (1976).

[8] For additional material on unit cost functions and the other theoretical results used in this section, see Diewert (1974) (1993).

[9] See Shephard (1953) or Diewert (1976; 120).

[10] This concept for a price index is due to Konüs (1924).

[11] See Diewert (1976; 120) for more details.

[12] In deriving (9), we also used $f(q^t) = \ f(q^t)\cdot q^t$ which follows from Euler's Theorem on homogeneous functions.

With the above economic preliminaries out of the way, we can derive a generalization of the Quadratic Identity, (4).

3. A Transformed Quadratic Identity

We now suppose that our aggregator function f(q) has the following *transformed quadratic functional form*:

(10) $g[f(q)] \equiv a_0 + \sum_{n=1}^{N} a_n\, h(q_n) + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\, h(q_i)h(q_j)$; $a_{ij} = a_{ji}$

where the a_n and the a_{ij} are constants and the functions g and h are continuous monotonic functions of one variable with nonzero derivatives. Later, we will specialize the general functional form f defined by (10) by choosing specific functions for g and h and we will place some restrictions on the coefficients a_n and a_{ij} so that the resulting f will be linearly homogeneous.

It is fairly obvious that (10) can be rewritten as a quadratic function of the type defined by (1) if we make some transformations of variables. Thus define:

(11) $z_n \equiv h(q_n)$; $\qquad n = 1,2,\ldots,N.$

Due to the assumed continuity and monotonicity of the function h, we can invert equations (11):

(12) $q_n = h^{-1}(z_n)$; $\qquad n = 1,2,\ldots,N.$

We rewrite the N equations in (12) in vector notation as follows:

(13) $q = H^{-1}(z)$

where $q \equiv (q_1,\ldots,q_N)$ and $z \equiv (z_1,\ldots,z_N)$. Now define the function of N variables F(z) by:

(14) $F(z) \equiv g[f\{H^{-1}(z)\}].$

Substituting (11)-(14) into (10) shows that the F defined by (14) is the quadratic function of z defined by (1).

We now need to express the first order partial derivatives of F, $F_n(z) \equiv \partial F(z)/\partial z_n$, in terms of f, g and h. First note that since $h'(q_n) \neq 0$ by assumption, we have

(15) $dh^{-1}(z_n)/dz_n = 1/h'(q_n)$; $\qquad n = 1,2,\ldots,N.$

Now differentiate (14) with respect to z_n:

(16) $F_n(z) = g'[f\{H^{-1}(z)\}]\, f_n\{H^{-1}(z)\}\, dh^{-1}(z_n)/dz_n$

$$= g'[f(q)]\, f_n(q)\, dh^{-1}(z_n)/dz_n \qquad \text{using (13)}$$
$$= g'[f(q)]\, f_n(q) / h'(q_n) \qquad \text{using (15).}$$

Now substitute (16) into (4) and we obtain the following identity:

$$(17)\quad g[f(q^1)] - g[f(q^0)] = \sum_{n=1}^{N} (1/2)[F_n(z^0) + F_n(z^1)][h(q_n^{\,1}) - h(q_n^{\,0})]$$
$$= \sum_{n=1}^{N} (1/2)[\{f_n(q^0)g'[f(q^0)]/h'(q_n^{\,0})\} + \{f_n(q^1)g'[f(q^1)]/h'(q_n^{\,1})\}][\, h(q_n^{\,1}) - h(q_n^{\,0})].$$

Equation (17) is our *generalized quadratic identity* and it holds as an identity for all functions f defined by (10).

To illustrate the usefulness of (17), let g and h be the natural logarithm functions; i.e., define:

$$(18)\quad g(y) \equiv \ln y \text{ and } h(y) \equiv \ln y.$$

Using $g'(y) = 1/y$ and $h'(y) = 1/y$, (17) becomes

$$(19)\quad \ln f(q^1) - \ln f(q^0) = \sum_{n=1}^{N} (1/2)[\{f_n(q^0)q_n^{\,0}/f(q^0)\} + \{f_n(q^1)q_n^{\,1}/f(q^1)\}][\ln q_n^{\,1} - \ln q_n^{\,0}]$$

and (10) becomes

$$(20)\quad \ln f(q) \equiv a_0 + \sum_{n=1}^{N} a_n \ln q_n + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \ln q_i \ln q_j\ ;\ \ a_{ij} = a_{ji}\ .$$

Note that the f(q) defined by (20) becomes the well known *translog aggregator function*.[13]

In order to make the f(q) defined by (20) linearly homogeneous, we require the following restrictions:

$$(21)\quad \sum_{n=1}^{N} a_n = 1\ ;\ \ \sum_{j=1}^{N} a_{ij} = 0\ ;\ i = 1,2,\ldots,N.$$

With the restrictions (21) imposed, f(q) defined by (20) is linearly homogeneous and we can apply Wold's identity (9), $f_n(q^t) = f(q^t)p_n^{\,t}/p^t\cdot q^t$, for $t = 0,1$ and $n = 1,2,\ldots,N$. Substituting these relations into (19) yields:

$$(22)\quad \ln [f(q^1)/f(q^0)] = (1/2) \sum_{n=1}^{N} [s_n^{\,0} + s_n^{\,1}] \ln [q_n^{\,1}/q_n^{\,0}]$$

where $s_n^{\,t} \equiv p_n^{\,t}q_n^{\,t}/p^t\cdot q^t$ is the share of period t expenditure on commodity n for $t = 0,1$ and $n = 1,2,\ldots,N$. The right hand side of (22) is the logarithm of the Törnqvist quantity index, $Q_T(p^0,p^1,q^0,q^1)$, and the left hand side of (22) is the logarithm of the true quantity index, $f(q^1)/f(q^0)$. Thus we have

---

[13] This functional form was introduced into the economics literature by Christensen, Jorgenson and Lau (1971) (1973).

(23) $f(q^1)/f(q^0) = Q_T(p^0,p^1,q^0,q^1)$.

Note that the right hand sides of (22) and (23) can be calculated using observable data.

The above algebra can be repeated for the translog unit cost function, which can be defined by (20), except that $c(p)$ replaces $f(q)$ and $\ln p_n$ replaces $\ln q_n$. The counterpart to (19) becomes

(24) $\ln c(p^1) - \ln c(p^0) = \sum_{n=1}^{N} (1/2)[\{c_n(p^0)p_n^0/c(p^0)\} + \{c_n(p^1)p_n^1/c(p^1)\}][\ln p_n^1 - \ln p_n^0]$.

Again, we assume that the restrictions (21) hold so that the translog unit cost function $c(p)$ is linearly homogeneous in the components of p. Now use Shephard's lemma (8), $c_n(p^t) = c(p^t)q_n^t/p^t\cdot q^t$, for $t = 0,1$ and $n = 1,2,\ldots,N$. Substituting these relations into (24) yields:

(25) $\ln [c(p^1)/c(p^0)] = (1/2) \sum_{n=1}^{N} [s_n^0 + s_n^1] \ln [p_n^1/p_n^0]$.

The right hand side of (25) is the logarithm of the Törnqvist price index, $P_T(p^0,p^1,q^0,q^1)$, and the left hand side of (25) is the logarithm of the true price index, $c(p^1)/c(p^0)$. Thus we have

(26) $c(p^1)/c(p^0) = P_T(p^0,p^1,q^0,q^1)$.

The exact index number results (23) and (26) illustrate the usefulness of the generalized quadratic identity (17) even though these results are not new.[14] In the following section, we provide some new applications of (17).

4. The Generalized Quadratic Identity and Mean of Order r Indexes

Recall the generalized quadratic functional form defined by (10) above. We now place the following restrictions on the coefficients a_n:

(27) $a_n = 0$; $\qquad\qquad$ $n = 0,1,\ldots,N$.

We also assume that the functions g and h which appear in the definition of f are defined as follows:

(28) $g(y) \equiv y^r$; $h(y) \equiv y^{1/2r}$; $r \neq 0$.

Using the restrictions (27) and (28), the function f defined by (10) becomes the following *quadratic mean of order r aggregator function*:

(29) $f(q) \equiv [\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\, q_i^{1/2r}\, q_j^{1/2r}]^{1/r}$; $a_{ij} = a_{ji}$.

[14] See Diewert (1976; 119-121).

It can be shown that f(q) defined by (29) is linearly homogeneous flexible functional form; that is, it can provide a second order approximation to an arbitrary twice continuously differentiable linearly homogeneous function.[15]

Substituting the restrictions and definitions (27) and (28) into the generalized quadratic identity (17) yields the following identity:

$$(30)\quad [f(q^1)]^r - [f(q^0)]^r = (1/2)\sum_{n=1}^N [\{f_n(q^0)r[f(q^0)]^{r-1}/[(r/2)(q_n^0)^{(r/2)-1}]\} + \{f_n(q^1)r[f(q^1)]^{r-1}/[(r/2)(q_n^1)^{(r/2)-1}]\}[(q_n^1)^{r/2} - (q_n^0)^{r/2}]$$

or

$$(31)\quad [f^1]^r - [f^0]^r = \sum_{n=1}^N [f_n^0\{f^0\}^{r-1}(q_n^0)^{1-r/2} + f_n^1\{f^1\}^{r-1}(q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}]$$

where we have simplified the notation by defining

$$(32)\quad f^0 \equiv f(q^0)\ ;\ f^1 \equiv f(q^1)\ ;\ f_n^t \equiv f_n(q^t) \equiv \partial f(q^t)/\partial q_n\ ;\ t = 0,1\ ;\ n = 1,2,\ldots,N.$$

Since the f(q) defined by (29) is linearly homogeneous, we may use Wold's identity (9) to replace the partial derivatives f_n^t by $p_n^t f^t/p^t\cdot q^t$. The notation will be simplified if we define the *period t normalized price for commodity n*, w_n^t, as follows:

$$\begin{aligned}(33)\quad w_n^t &\equiv p_n^t/p^t\cdot q^t && t = 0,1\ ;\ n = 1,2,\ldots,N\\ &= [1/f(q^t)]\,\partial f(q^t)/\partial q_n && \text{using Wold's identity, (9)}\\ &= \partial \ln f(q^t)/\partial q_n &&\\ &= f_n^t/f^t && \text{using the notation in (32).}\end{aligned}$$

Thus w_n^t is the period t price for commodity n, p_n^t, divided by period t expenditure on all commodities in the aggregate, $p^t\cdot q^t \equiv \sum_{n=1}^N p_n^t q_n^t$. Using Wold's identity, the normalized price w_n^t is equal to the logarithmic derivative of the aggregator function with respect to commodity n evaluated at the period t data, $\partial \ln f(q^t)/\partial q_n$. Making use of (33), (31) can be rewritten as follows:

$$(34)\quad [f^1]^r - [f^0]^r = \sum_{n=1}^N [w_n^0\{f^0\}^r(q_n^0)^{1-r/2} + w_n^1\{f^1\}^r(q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}].$$

Now divide both sides of (34) through by $[f^0]^r$ to obtain:

$$(35)\quad [f^1/f^0]^r - 1 = \sum_{n=1}^N [w_n^0(q_n^0)^{1-r/2} + w_n^1\{f^1/f^0\}^r(q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}].$$

Now f^1/f^0 is the true quantity index, $Q_r \equiv f(q^1)/f(q^0)$. Replace f^1/f^0 in (35) by Q_r and solve the resulting equation for Q_r. We obtain the following solution:

[15] See Diewert (1976; 130). Färe and Sung (1986) showed that the translog case considered earlier and the present normalized quadratic function are the *only* special cases of the family of generalized quadratic functions that are also linearly homogeneous.

$$(36)\quad Q_r = [\sum_{n=1}^{N} (q_n^1/q_n^0)^{r/2}\, s_n^0\,]^{1/r}\, [\sum_{n=1}^{N} (q_n^1/q_n^0)^{-r/2}\, s_n^1]^{-1/r}$$

where s_n^t is the period t expenditure share for commodity n; i.e.,

$$(37)\quad s_n^t \equiv p_n^t q_n^t / p^t \cdot q^t\ ; \qquad t = 0,1\ ;\ n = 1,2,\ldots,N$$
$$= w_n^t q_n^t \qquad \text{using definitions (33).}$$

The index number formula on the right hand side of (36) depends only on the observed prices and quantities pertaining to the two periods under consideration and it is equal to the *quadratic mean of order r quantity index* defined by Diewert (1976; 130). The above results show that it is *exactly* equal to $f(q^1)/f(q^0)$ where f is the *quadratic mean of order r aggregator function* defined by (29). Thus we have used the generalized quadratic identity (17) in order to establish this exactness result.

The above algebra for quantity indexes has its counterpart for price indexes as we now show. Define the *quadratic mean of order r unit cost function* c(p) by:

$$(38)\quad c(p) \equiv [\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\, p_i^{1/2r}\, p_j^{1/2r}]^{1/r}\ ;\ a_{ij} = a_{ji}\,.$$

Define the *period t normalized quantity for commodity n*, v_n^t, as follows:

$$(39)\quad v_n^t \equiv q_n^t / p^t \cdot q^t \qquad t = 0,1\ ;\ n = 1,2,\ldots,N$$
$$= [\partial c(p^t)/\partial p_n]/c(p^t) \qquad \text{using Shephard's lemma, (8)}$$
$$= \partial \ln c(p^t)/\partial p_n\,.$$

Let the level of unit cost in period 0 be $c^0 \equiv c(p^0)$ and the level of unit cost in period 1 be $c^1 \equiv c(p^1)$. The counterpart to (35) is now

$$(40)\quad [c^1/c^0]^r - 1 = \sum_{n=1}^{N} [v_n^0\, (p_n^0)^{1-r/2} + v_n^1 \{c^1/c^0\}^r (p_n^1)^{1-r/2}][(p_n^1)^{r/2} - (p_n^0)^{r/2}].$$

Note that $c^1/c^0 \equiv c(p^1)/c(p^0) \equiv P_r$ is the true price index that corresponds to the unit cost function defined by (38). Now replace c^1/c^0 in equation (40) by P_r and solve the resulting equation for P_r. We obtain the following solution:

$$(41)\quad P_r = [\sum_{n=1}^{N} (p_n^1/p_n^0)^{r/2}\, s_n^0\,]^{1/r}\, [\sum_{n=1}^{N} (p_n^1/p_n^0)^{-r/2}\, s_n^1]^{-1/r}$$

where s_n^t is the period t expenditure share for commodity n defined earlier by (37). The index number formula on the right hand side of (41) depends only on the observed prices and quantities pertaining to the two periods under consideration and it is equal to the *quadratic mean of order r price index* defined by Diewert (1976; 131). The above results show that it is *exactly* equal to $c(p^1)/c(p^0)$ where c is the *quadratic mean of order r unit cost function* defined by (38). Thus again we have used the generalized quadratic identity (17) in order to establish this exactness result.

In the following two sections, we examine equations (35) and (40) more closely for the special cases when r = 1 or 2.

5. Additive Percentage Change Decompositions for the Fisher Ideal Indexes

It can be verified that when r = 2, Q_2 defined by (36) turns out to equal the *Fisher (1922) ideal quantity index* Q_F; i.e., we have

(42) $Q_2 = Q_F(p^0,p^1,q^0,q^1) \equiv [\{p^0\cdot q^1/p^0\cdot q^0\}\{p^1\cdot q^1/p^1\cdot q^0\}]^{1/2}$.

Using (34) when r = 2, we have the following decomposition:[16]

(43) $[f^{\,1}]^2 - [f^{\,0}]^2 = \sum_{n=1}^{N} [w_n^{\,0}\{f^{\,0}\}^2 + w_n^{\,1}\{f^{\,1}\}^2][q_n^{\,1} - q_n^{\,0}]$

where the normalized prices w_n^t are defined by (33). From elementary algebra, we have:

(44) $[f^{\,1}]^2 - [f^{\,0}]^2 = [f^{\,1} - f^{\,0}]\,[f^{\,1} + f^{\,0}]$.

Now divide both sides of (43) by $f^{\,1} + f^{\,0}$. Using (44), the resulting equation becomes:

(45) $f^{\,1} - f^{\,0} = f^{\,0} \sum_{n=1}^{N} \{(f^{\,0}/[f^{\,0} + f^{\,1}])w_n^{\,0} + (f^{\,1}/f^{\,0})\{(f^{\,1}/[f^{\,0} + f^{\,1}])w_n^{\,1}\}\{q_n^{\,1} - q_n^{\,0}\}$.

Divide both sides of (45) by $f^{\,0}$ and using $Q_F = f^{\,1}/f^{\,0}$, we have the following *additive percentage change decomposition for the Fisher ideal quantity index*:[17]

(46) $Q_F - 1 = \sum_{n=1}^{N} \{(1/[1 + Q_F])w_n^{\,0} + (Q_F/[1 + Q_F])\, Q_F\, w_n^{\,1}\}\{q_n^{\,1} - q_n^{\,0}\}$.

In the above decomposition, the term in front of the change in quantity n going from period 0 to 1, Q_{Fn}, *the nth percentage change quantity weight*, is defined as follows:

(47) $Q_{Fn} \equiv (1/[1 + Q_F])w_n^{\,0} + (Q_F/[1 + Q_F])\, Q_F\, w_n^{\,1}$.

Note that the nth percentage change quantity weight is almost a weighted average (with weights $(1/[1 + Q_F])$ and $(Q_F/[1 + Q_F])$ which sum to unity) of the two normalized prices for commodity n in the two periods under consideration, $w_n^t \equiv p_n^t/p^t\cdot q^t$ for t = 0,1. However, the period 1 normalized price w_n^1 gets an *extra* weighting factor equal to Q_F, the value of the Fisher quantity index going from period 0 to 1. If $Q_F = 1$, then Q_{Fn} is equal to the arithmetic average of the normalized prices for commodity n, $(1/2)w_n^0 + (1/2)w_n^1$.

In a manner analogous to the derivation of (46), we can obtain the following *additive percentage change decomposition for the Fisher ideal price index*:

(48) $P_F - 1 = \sum_{n=1}^{N} \{(1/[1 + P_F])v_n^{\,0} + (P_F/[1 + P_F])\, P_F\, v_n^{\,1}\}\{p_n^{\,1} - p_n^{\,0}\}$

[16] This decomposition was used already by Reinsdorf, Diewert and Ehemann (2000).

[17] If we solve equation (46) for Q_F, we obtain the Fisher ideal index defined by (42) as the solution. This shows that (46) is an identity, which is valid for all p^0,p^1,q^0,q^1.

where the *Fisher ideal price index* P_F is defined as follows:

$$(49)\quad P_F(p^0,p^1,q^0,q^1) \equiv [\{p^1\cdot q^0/p^0\cdot q^0\}\{p^1\cdot q^1/p^0\cdot q^1\}]^{1/2}$$
$$= p^1\cdot q^1/p^0\cdot q^0\ Q_F(p^0,p^1,q^0,q^1)$$

where Q_F is the Fisher ideal quantity index defined earlier by (42). In the decomposition (48), the term in front of the change in price n going from period 0 to 1, P_{Fn}, *the nth percentage change price weight*, is defined as follows:

$$(50)\quad P_{Fn} \equiv (1/[1 + P_F])v_n^0 + (P_F/[1 + P_F])\ P_F\ v_n^1$$

where the *normalized quantities*, v_n^t are equal to $q_n^t/p^t\cdot q^t$ for $t = 0,1$.

It should be noted that the concept of a price or quantity index number formula having an *additive percentage change decomposition* is not quite the same as an index number formula having the property of *additivity*. We now explain the difference.

A price index, $P(p^0,p^1,q^0,q^1)$, is said to be *additive* if it can be written as follows:

$$(51)\quad P(p^0,p^1,q^0,q^1) = \sum_{n=1}^N q_n^* p_n^1 / \sum_{n=1}^N q_n^* p_n^0$$

where the "quantity" weights q_n^* are *usually* taken to be some sort of average of the base and current period quantities for commodity n, q_n^0 and q_n^1. However, in principle, more complicated quantity weighting could be used: the important factor in the definition of additivity given by (51) is that the quantity weights be the *same* in the numerator and the denominator of the right hand side of (51).

In a similar manner, a quantity index, $Q(p^0,p^1,q^0,q^1)$, is said to be *additive* if it can be written as follows:

$$(52)\quad Q(p^0,p^1,q^0,q^1) = \sum_{n=1}^N p_n^* q_n^1 / \sum_{n=1}^N p_n^* q_n^0$$

where the "price" weights p_n^* are *usually* taken to be some sort of average of the base and current period prices for commodity n, p_n^0 and p_n^1. However, in principle, more complicated price weighting could be used: as before, the important factor in the definition of additivity given by (52) is that the price weights be the *same* in the numerator and the denominator of the right hand side of (52).

It is straightforward to show that additive price and quantity indexes have additive percentage change decompositions. For example, suppose we have an additive quantity index of the type defined by (52) above. Then we have:

$$(53)\quad Q(p^0,p^1,q^0,q^1) - 1 = [\sum_{n=1}^N p_n^* q_n^1 / \sum_{n=1}^N p_n^* q_n^0] - 1 \qquad \text{using (52)}$$
$$= [\sum_{n=1}^N p_n^* q_n^1 / \sum_{n=1}^N p_n^* q_n^0] - [\sum_{n=1}^N p_n^* q_n^0 / \sum_{n=1}^N p_n^* q_n^0]$$
$$= [\sum_{n=1}^N p_n^* \{q_n^1 - q_n^0\}] / \sum_{n=1}^N p_n^* q_n^0$$

$$= \sum_{n=1}^{N} Q_n\{q_n^1 - q_n^0\}$$

where the *nth percentage change quantity weight* Q_n is defined as

(54) $Q_n \equiv p_n^*/ \sum_{n=1}^{N} p_n^* q_n^0$; $\qquad$ n = 1,…,N.

Thus we can always find an additive percentage change decomposition for an additive price or quantity index. However, it is not always possible to go from an additive percentage change decomposition to a corresponding additive index number formula.[18]

Unfortunately, the additive percentage change decompositions (46) and (48) that we obtained for the Fisher quantity and price indexes are not unique. For example, Van Ijzeren (1987;6) chose the following values for the quantity weights q_n^* which appear in (51) above:

(55) $q_n^* \equiv (1/2)q_n^0 + (1/2)q_n^1/Q_F(p^0,p^1,q^0,q^1)$; $\qquad$ n = 1,2,…,N

where Q_F is the Fisher quantity index defined by (42). Thus the reference quantity for commodity n in formula (51), q_n^*, is chosen to be the arithmetic mean of the period 0 and period 1 observed use of commodity n, except that the period 1 use, q_n^1, is deflated by the Fisher quantity index, Q_F, for the entire group of commodities in the aggregate. To show that the resulting price index defined by (51) is the Fisher ideal price index, replace Q_F in (55) by $p^1\cdot q^1/p^0\cdot q^0\, P$ if necessary. Now solve (51) using the weights defined by (55) for P; i.e., solve the resulting equation for P:

(56) $P = \sum_{n=1}^{N} (1/2)[q_n^0 + q_n^1/Q\}]p_n^1 / \sum_{n=1}^{N} (1/2)[q_n^0 + q_n^1/Q]p_n^0$

$= [p^1\cdot q^0 + p^1\cdot q^1/Q] / [p^0\cdot q^0 + p^0\cdot q^1/Q]$ $\qquad$ or

$PQp^0\cdot q^0 + p^0\cdot q^1\, P = Qp^1\cdot q^0 + p^1\cdot q^1$ $\qquad$ or

$[p^1\cdot q^1/p^0\cdot q^0]\, p^0\cdot q^0 + p^0\cdot q^1\, P = Qp^1\cdot q^0 + p^1\cdot q^1$ $\qquad$ using $PQ = [p^1\cdot q^1/p^0\cdot q^0]$ or

$p^0\cdot q^1\, P = Qp^1\cdot q^0$ $\qquad$ canceling terms or

$p^0\cdot q^1\, P = [p^1\cdot q^1/p^0\cdot q^0\, P]p^1\cdot q^0$ $\qquad$ using $Q = p^1\cdot q^1/p^0\cdot q^0\, P$ or

$P^2 = [p^1\cdot q^1/p^0\cdot q^1][\, p^1\cdot q^0/p^0\cdot q^0]$ $\qquad$ or

(57) $P = \{[p^1\cdot q^1/p^0\cdot q^1][\, p^1\cdot q^0/p^0\cdot q^0]\}^{1/2} \equiv P_F$.

Thus (51) and (55) provide an exact additive representation for the Fisher ideal price index.

In a similar fashion, Van Ijzeren (1987;6) showed that if we choose the following values for the price weights p_n^* which appear in (52) above:

[18] It does not seem to be possible to go from the additive percentage change decomposition for the Fisher quantity index given by (46) to a corresponding additive representation of the form defined by (52).

(58) $p_n^* \equiv (1/2)p_n^0 + (1/2)p_n^1/P_F(p^0,p^1,q^0,q^1)$; $\qquad n = 1,2,\ldots,N$

where P_F is the Fisher price index defined by (49), then (52) and (58) provide an exact additive representation for the Fisher ideal quantity index.

Since an additive representation for index number formula implies an additive percentage change decomposition for the formula, we see that our additive percentage change decompositions for the Fisher ideal price and quantity indexes given by (48) and (46) above are not unique.[19]

In retrospect, it is not surprising that additive percentage change decompositions of any index number formula are not unique (unless the decomposition has to satisfy further properties). To see this, look at the right hand side of equation (52), which is homogeneous of degree 0 in $p_1^*,\ldots,p_N^*$. Thus given an index value Q, we can never determine the scale of the p_n^*. Hence, let us impose a normalization on the p_n^*, such as:

(59) $\sum_{n=1}^N p_n^*q_n^0 = 1.$

Using (59), equation (52), which defines an additive representation for the quantity index Q, can be rewritten as follows:

(60) $\sum_{n=1}^N p_n^*q_n^1 = Q.$

Equations (59) and (60) can be regarded as two simultaneous linear equations in the N unknowns, $p_1^*,\ldots,p_N^*$. Obviously, as soon as N exceeds 2, there will be an infinite number of solutions to (59) and (60) in general. Thus the quest for *unique* additive representations or *unique* additive percentage change decompositions of an index number formula is doomed to failure. Hence any particular additive percentage change decomposition needs to be justified on axiomatic grounds or on its economic interpretation. We will return to this topic after the following section.

In the following section, we examine equations (35) and (40) for the special case when r equals 1.

6. Additive Percentage Change Decompositions for the Implicit Walsh Indexes

Our goal in this section is to provide some additive percentage change decompositions for some indexes defined by Walsh.

---

[19] In fact, two additional additive percentage change decompositions for the Fisher indexes may be found in Reinsdorf, Diewert and Ehemann (2000). The first of these two decompositions turns out to be equivalent to the decomposition of Van Ijzeren (1987; 6), which was also independently derived by Dikhanov (1997). The Van Ijzeren decomposition is currently being used by Bureau of Economic Analysis; see Moulton and Seskin (1999; 16) and Ehemann, Katz and Moulton (2000).

Walsh(1901; 398) (1921; 97) defined the following very useful price index:[20]

(61) $P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N [q_n^0 q_n^1]^{1/2} p_n^1 / \sum_{n=1}^N [q_n^0 q_n^1]^{1/2} p_n^0$.

Thus the *Walsh price index* is an additive price index of the type defined by (51) where the quantity weights q_n^* are equal to the geometric means of the period 0 and 1 consumption of commodity n, $[q_n^0 q_n^1]^{1/2}$. As indicated in the previous section, the Walsh price index necessarily has an additive percentage change decomposition.

Using the fact that the price index times the quantity index should equal the value ratio for the two periods under consideration; i.e., using[21]

(62) $P_W(p^0,p^1,q^0,q^1)\, Q_W^*(p^0,p^1,q^0,q^1) = p^1\cdot q^1/p^0\cdot q^0$,

we can define the *implicit Walsh quantity index* Q_W^* that corresponds to P_W defined by (61) as follows:

(63) $Q_W^*(p^0,p^1,q^0,q^1) \equiv [p^1\cdot q^1/p^0\cdot q^0]/P_W(p^0,p^1,q^0,q^1)$
$= \{\sum_{n=1}^N [q_n^0 q_n^1]^{1/2} p_n^0/p^0\cdot q^0\}/\{\sum_{n=1}^N [q_n^0 q_n^1]^{1/2} p_n^1/ p^1\cdot q^1\}$ using (61)
$= \{\sum_{n=1}^N [q_n^1/q_n^0]^{1/2} p_n^0 q_n^0/p^0\cdot q^0\}/\{\sum_{n=1}^N [q_n^0/q_n^1]^{1/2} p_n^1 q_n^1/ p^1\cdot q^1\}$
$= \{\sum_{n=1}^N [q_n^1/q_n^0]^{1/2} s_n^0\}/\{\sum_{n=1}^N [q_n^0/q_n^1]^{1/2} s_n^1\}$ using (37)
$= Q_1(p^0,p^1,q^0,q^1)$

where Q_1 is a *quadratic mean of order r quantity index* defined by (36) when r = 1. Thus the Walsh implicit quantity index, Q_W^*, is equal to a special case of the quadratic mean of order quantity indexes defined earlier.

It is not at all obvious what an additive percentage change decomposition for the implicit Walsh quantity index would look like. However, using the decomposition (35) for r = 1 yields the following equation:

(64) $Q_1 - 1 = \sum_{n=1}^N [w_n^0 (q_n^0)^{1/2} + w_n^1\{Q_1\}(q_n^1)^{1/2}][(q_n^1)^{1/2} - (q_n^0)^{1/2}]$

where Q_1 is defined by (63). Now multiply the numerator and the denominator of the nth term on the right hand side of (64) by $(q_n^1)^{1/2} + (q_n^1)^{1/2}$ for n = 1,…,N. The resulting equation is:

(65) $Q_1 - 1 = \sum_{n=1}^N [w_n^0 \{(q_n^0)^{1/2}/[(q_n^1)^{1/2} + (q_n^1)^{1/2}]\}$
$+ w_n^1 Q_1\{(q_n^1)^{1/2}/[(q_n^1)^{1/2} + (q_n^1)^{1/2}]\}[q_n^1 - q_n^0]$
$= \sum_{n=1}^N Q_{1n} [q_n^1 - q_n^0]$

[20] Diewert (2000) made a case for this index being the "best" pure price or fixed basket type index. The Australian statistician Knibbs (1924; 43-44) was perhaps the first to define the class of fixed basket type indexes, which he called unequivocal indexes.
[21] See (7) above. Fisher (1911) was the first to suggest that that the product of the price and quantity indexes should equal the value ratio between the two periods under consideration.

where the *nth Walsh percentage change quantity weight* Q_{1n} is defined as

(66) $Q_{1n} \equiv w_n^0 \{(q_n^0)^{1/2}/[(q_n^1)^{1/2} + (q_n^1)^{1/2}]\} + w_n^1 Q_1\{(q_n^1)^{1/2}/[(q_n^1)^{1/2} + (q_n^1)^{1/2}]\}$.

Note that the nth percentage change quantity weight is almost a weighted average (with weights $(q_n^0)^{1/2}/[(q_n^1)^{1/2} + (q_n^1)^{1/2}]$ and $(q_n^1)^{1/2}/[(q_n^1)^{1/2} + (q_n^1)^{1/2}]$ which sum to unity) of the two normalized prices for commodity n in the two periods under consideration, $w_n^t \equiv p_n^t / p^t \cdot q^t$ for t = 0,1. However, as was the case with the Fisher decomposition defined earlier by (46) and (47), the period 1 normalized price w_n^1 gets an *extra* weighting factor equal to Q_1, the value of the Walsh implicit quantity index going from period 0 to 1.

The counterpart to the Walsh price index defined by (61) above is the *Walsh* (1921; 103) *quantity index* Q_W defined as follows:[22]

(67) $Q_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N [p_n^0 p_n^1]^{1/2} q_n^1 / \sum_{n=1}^N [p_n^0 p_n^1]^{1/2} q_n^0$.

It is easy to see that the Walsh quantity index has the *additive* form defined by (52) where the nth price weight p_n^* is the geometric mean of the period 0 and 1 prices for commodity n, $[p_n^0 p_n^1]^{1/2}$. Thus the Walsh quantity index also has an additive percentage change decomposition; recall (53) and (54) above.

The *Walsh* (1921; 103) *implicit price index* that corresponds to the Walsh quantity index Q_W defined by (67) is defined as follows:

$$
\begin{aligned}
(68)\quad P_W^*(p^0,p^1,q^0,q^1) &\equiv [p^1 \cdot q^1/p^0 \cdot q^0]/Q_W(p^0,p^1,q^0,q^1) \\
&= \{\textstyle\sum_{n=1}^N [p_n^0 p_n^1]^{1/2} q_n^0/p^0 \cdot q^0\}/\{\sum_{n=1}^N [p_n^0 p_n^1]^{1/2} q_n^1/ p^1 \cdot q^1\} && \text{using (67)} \\
&= \{\textstyle\sum_{n=1}^N [p_n^1/p_n^0]^{1/2} p_n^0 q_n^0/p^0 \cdot q^0\}/\{\sum_{n=1}^N [p_n^0/p_n^1]^{1/2} p_n^1 q_n^1/ p^1 \cdot q^1\} \\
&= \{\textstyle\sum_{n=1}^N [p_n^1/p_n^0]^{1/2} s_n^0\}/\{\sum_{n=1}^N [p_n^0/p_n^1]^{1/2} s_n^1\} && \text{using (37)} \\
&= P_1(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where P_1 is a *quadratic mean of order r price index* defined by (42) when r = 1. Thus the Walsh implicit price index, P_W^*, is equal to a special case of the quadratic mean of order quantity indexes defined earlier.

We can repeat the algebra associated with (64) and (65) above using the decomposition (40) in place of (35) to show that the Walsh implicit price index has the following additive percentage change decomposition:

(69) $P_1 - 1 = \sum_{n=1}^N P_{1n} [p_n^1 - p_n^0]$

where the *nth Walsh percentage change price weight* P_{1n} is defined as

(70) $P_{1n} \equiv v_n^0 \{(p_n^0)^{1/2}/[(p_n^1)^{1/2} + (p_n^1)^{1/2}]\} + v_n^1 P_1\{(p_n^1)^{1/2}/[(p_n^1)^{1/2} + (p_n^1)^{1/2}]\}$.

[22] The Walsh quantity index is a special case of Knibb's (1924; 43-44) class of *unequivocal quantity indexes*. See Diewert (2000) for further discussion.

Note that the nth percentage change price weight is almost a weighted average (with weights $(p_n^0)^{1/2}/[(p_n^1)^{1/2} + (p_n^1)^{1/2}]$ and $(p_n^1)^{1/2}/[(p_n^1)^{1/2} + (p_n^1)^{1/2}]$ which sum to unity) of the two normalized quantities for commodity n in the two periods under consideration, $v_n^t \equiv q_n^t / p^t \cdot q^t$ for t = 0,1. However, as was the case with the Fisher decomposition defined earlier by (48), the period 1 normalized quantity v_n^1 gets an *extra* weighting factor equal to P_1, the value of the Walsh implicit price index going from period 0 to 1.

The results in this section show that all four of the Walsh price and quantity indexes have additive percentage change decompositions. In the following section, we will attempt to provide economic interpretations for the terms in two of these additive decompositions.

7. Economic Interpretations for Some Additive Percentage Change Decompositions

Given that in general an infinite number of additive percentage change decompositions are possible for any given price or quantity index, it will be useful to find decompositions such that each term in the decomposition has an economic interpretation. In this section, we shall show how this can be done for some of the most commonly used superlative index number formulae.[23]

We first need to provide an exact interpretation for each of the N terms on the right hand side of the quadratic identity (4) above. Let F(z) be the quadratic function defined by (1) and consider a change in the vector z from the base period situation $z^0 \equiv (z_1^0,z_2^0,\ldots,z_N^0)$ to $(z_1^1,z_2^0,\ldots,z_N^0)$; i.e., only the first component of z changes from the base period value z_1^0 to the period 1 value z_1^1. Then since $F(z_1,z_2,\ldots,z_N)$ is quadratic in z_1, we can apply the quadratic identity (4) to this change and get the following equation:

$$
\begin{aligned}
(71)\quad & F(z_1^1,z_2^0,\ldots,z_N^0) - F(z_1^0,z_2^0,\ldots,z_N^0) \\
&= (1/2)[F_1(z_1^0,z_2^0,\ldots,z_N^0) + F_1(z_1^1,z_2^0,\ldots,z_N^0)][z_1^1 - z_1^0] \\
&= (1/2)[a_1+2a_{11}z_1^0+\textstyle\sum_{i=2}^N 2a_{1i}z_i^0 + a_1+2a_{11}z_1^1+\sum_{i=2}^N 2a_{1i}z_i^0][z_1^1 - z_1^0] \\
&\qquad\qquad\qquad\qquad \text{partially differentiating the F defined by (1)} \\
&= (1/2)[F_1(z_1^0,z_2^0,\ldots,z_N^0)+2a_{11}(z_1^1 - z_1^0)+F_1(z_1^0,z_2^0,\ldots,z_N^0)][z_1^1 - z_1^0] \\
&= [F_1(z_1^0,z_2^0,\ldots,z_N^0)+a_{11}(z_1^1 - z_1^0)][z_1^1 - z_1^0].
\end{aligned}
$$

Now consider a change in z from $(z_1^0,z_2^1,\ldots,z_N^1)$ to $z^1 \equiv (z_1^1,z_2^1,\ldots,z_N^1)$. In a manner analogous to our derivation of (71), we can show that

$$
\begin{aligned}
(72)\quad & F(z_1^1,z_2^1,\ldots,z_N^1) - F(z_1^0,z_2^1,\ldots,z_N^1) \\
&= (1/2)[F_1(z_1^0,z_2^1,\ldots,z_N^1) + F_1(z_1^1,z_2^1,\ldots,z_N^1)][z_1^1 - z_1^0] \\
&= (1/2)[a_1+2a_{11}z_1^0+\textstyle\sum_{i=2}^N 2a_{1i}z_i^1 + a_1+2a_{11}z_1^1+\sum_{i=2}^N 2a_{1i}z_i^1][z_1^1 - z_1^0] \\
&\qquad\qquad\qquad\qquad \text{partially differentiating the F defined by (1)} \\
&= (1/2)[F_1(z_1^1,z_2^1,\ldots,z_N^1)+2a_{11}(z_1^0 - z_1^1)+F_1(z_1^1,z_2^1,\ldots,z_N^1)][z_1^1 - z_1^0] \\
&= [F_1(z_1^1,z_2^1,\ldots,z_N^1) - a_{11}(z_1^1 - z_1^0)][z_1^1 - z_1^0].
\end{aligned}
$$

---

[23] For the Törnqvist price and quantity indexes, we will obtain multiplicative decompositions rather than additive ones.

Finally, take the arithmetic average of equations (71) and (72) and we obtain the following exact identity:

$$(73)\quad (1/2)[F(z_1^1,z_2^0,\ldots,z_N^0) - F(z_1^0,z_2^0,\ldots,z_N^0)]+(1/2)[F(z_1^1,z_2^1,\ldots,z_N^1) - F(z_1^0,z_2^1,\ldots,z_N^1)]$$
$$= (1/2)[F_1(z_1^0,z_2^0,\ldots,z_N^0) + F_1(z_1^1,z_2^1,\ldots,z_N^1)][z_1^1 - z_1^0].$$

Note that the right hand side of (73) is the first term on the right hand side of the quadratic identity (4). Thus this first term is equal to the arithmetic average of two differences in the level of F(z) where only the first component of the z vector changes in each of these two differences.

We define the left hand side of (73) as the *first difference effect*, δ_1. In general, define the *nth difference effect*, δ_n, as follows:

$$(74)\quad \delta_n \equiv (1/2)[F(z_1^0,\ldots,z_{n-1}^0,z_n^1,z_{n+1}^0\ldots,z_N^0) - F(z_1^0,z_2^0,\ldots,z_N^0)]$$
$$+ (1/2)[F(z_1^1,z_2^1,\ldots,z_N^1) - F(z_1^1,\ldots z_{n-1}^1,z_n^0,z_{n+1}^1\ldots,z_N^1)] ; \qquad n = 1,2,\ldots,N.$$

Thus δ_n is the arithmetic average of two hypothetical changes in F(z) where in the first (second) change, only the nth component changes from its period 0 level of z_n^0 to its period 1 level z_n^1 and all other components of z are held constant at their period 0 (1) levels. In a manner analogous to our derivation of (73), we can show that δ_n is equal to the nth term on the right hand side of the quadratic identity (4); i.e., we have:

$$(75)\quad \delta_n = (1/2)[F_n(z_1^0,z_2^0,\ldots,z_N^0) + F_n(z_1^1,z_2^1,\ldots,z_N^1)][z_n^1 - z_n^0] \qquad n = 1,2,\ldots,N.$$

We now have to translate equations (74) and (75) into our generalized quadratic identity framework. If f(q) is defined by (10), it is straightforward to show that the counterpart to (75) is

$$(76)\quad \delta_n = (1/2)[\{f_n(q^0)g'[f(q^0)]/h'(q_n^0)\} + \{f_n(q^1)g'[f(q^1)]/h'(q_n^1)\}][h(q_n^1) - h(q_n^0)] ;$$
$$n = 1,2,\ldots,N$$

where δ_n is now defined as follows:

$$(77)\quad \delta_n \equiv (1/2)\{g[f(q_1^0,\ldots,q_{n-1}^0,q_n^1,q_{n+1}^0\ldots,q_N^0)] - g[f(q_1^0,q_2^0,\ldots,q_N^0)]\}$$
$$+ (1/2)\{g[f(q_1^1,q_2^1,\ldots,q_N^1)] - g[f(q_1^1,\ldots q_{n-1}^1,q_n^0,q_{n+1}^1\ldots,q_N^1)]\} ; \quad n = 1,2,\ldots,N.$$

Note that the right hand side of (76) is the nth term in our generalized quadratic identity and (77) gives an economic interpretation for this term in terms of differences in g[f(q)] where only the nth component of q changes. *Thus each of the N terms on the right hand side of the generalized quadratic identity (17) has an economic interpretation as an average of two finite differences in the level of our transformed aggregator function g[f(q)] where only one component of q changes in each of the finite differences.*

We now specialize (76) and (77) by considering specific functions for g and h.

The first special case that we consider is the case where g and h are the natural logarithm functions (recall (18) above), which gave rise to the translog aggregator function defined by (20) and (21). In this case, the generalized quadratic identity (17) became (22). Thus we have:

$$(78)\quad \ln [f(q^1)/f(q^0)] = \sum_{n=1}^N (1/2)[s_n^0 + s_n^1] \ln [q_n^1/q_n^0] = \sum_{n=1}^N \delta_n$$

where δ_n is defined by (77) where g is the logarithm function in this special case.

It is useful to introduce some additional notation at this point. Define *the base period nth quantity effect* σ_n^0 as the relative change in the aggregate going from the base period quantities q^0 to new quantities where we only change q_n to the period 1 level, q_n^1; i.e., define σ_n^0 as follows:

$$(79)\quad \sigma_n^0 \equiv f(q_1^0,\ldots,q_{n-1}^0,q_n^1,q_{n+1}^0\ldots,q_N^0)/f(q_1^0,q_2^0,\ldots,q_N^0)\ ; \qquad n = 1,2,\ldots,N.$$

Define *the current period nth quantity effect* σ_n^1 as the relative change in the aggregate going to the current period quantities q^1 from quantities where all quantities are at their period 1 levels except q_n is equal to the period 0 level, q_n^0; i.e., define σ_n^1 as follows:

$$(80)\quad \sigma_n^1 \equiv f(q_1^1,q_2^1,\ldots,q_N^1)/f(q_1^1,\ldots q_{n-1}^1,q_n^0,q_{n+1}^1\ldots,q_N^1)\ ; \qquad n = 1,2,\ldots,N.$$

Finally, define *the nth quantity effect* c_n as the geometric mean of the base and current period quantity effects defined by (79) and (80); i.e., define

$$(81)\quad c_n \equiv [\sigma_n^0\sigma_n^1]^{1/2}\ ; \qquad n = 1,2,\ldots,N.$$

Using this new notation and exponentiating both sides of (78), we obtain the following decomposition for the Törnqvist quantity index, $Q_T(p^0,p^1,q^0,q^1)$ (recall (23) above):[24]

$$\begin{aligned}(82)\quad f(q^1)/f(q^0) &= Q_T(p^0,p^1,q^0,q^1) \\ &= \prod_{n=1}^N \exp[\delta_n] \qquad \text{where } \exp[y] \equiv e^y \\ &= \prod_{n=1}^N [\sigma_n^0\sigma_n^1]^{1/2} \\ &= \prod_{n=1}^N c_n.\end{aligned}$$

Thus we have an *exact multiplicative decomposition* of the Törnqvist quantity index Q_T into a product of N quantity effects, $\prod_{n=1}^N c_n$, where each quantity effect is a quantity index which shows the effect of changing just the nth quantity from q_n^0 to q_n^1; see (79) to (81) above.

[24] For similar decompositions in the profit or revenue function context, see Diewert and Morrison (1986; 666-667) and Kohli (1990).

The same algebra works for a multiplicative decomposition for the Törnqvist price index P_T defined earlier by (25) and (26). Again, we introduce some additional notation in order to define the terms in the decomposition. Define *the base period nth price effect* ρ_n^0 as the relative change in the aggregate going from the base period prices p^0 to new prices where we only change p_n to the period 1 level, p_n^1; i.e., define ρ_n^0 as follows:

(83) $\rho_n^0 \equiv c(p_1^0,\ldots,p_{n-1}^0,p_n^1,p_{n+1}^0\ldots,p_N^0)/c(p_1^0,p_2^0,\ldots,p_N^0)$; $\quad$ n = 1,2,…,N

where c(p) is the translog unit cost function defined in section 3 above. Define *the current period nth price effect* ρ_n^1 as the relative change in the aggregate going to the current period prices p^1 from prices where all prices are at their period 1 levels except p_n is equal to the period 0 level, p_n^0; i.e., define ρ_n^1 as follows:

(84) $\rho_n^1 \equiv c(p_1^1,p_2^1,\ldots,p_N^1)/c(p_1^1,\ldots p_{n-1}^1,p_n^0,p_{n+1}^1\ldots,p_N^1)$; $\quad$ n = 1,2,…,N.

Finally, define *the nth price effect* b_n as the geometric mean of the base and current period quantity effects defined by (83) and (84); i.e., define

(85) $b_n \equiv [\rho_n^0\rho_n^1]^{1/2}$; $\quad$ n = 1,2,…,N.

Using this new notation, we obtain the following decomposition for the Törnqvist price index, $P_T(p^0,p^1,q^0,q^1)$ (recall (26) above):[25]

(86)
$$\begin{aligned} c(p^1)/c(p^0) &= P_T(p^0,p^1,q^0,q^1) \\ &= \prod_{n=1}^N \exp\{(1/2)\,[s_n^0 + s_n^1] \ln [p_n^1/p_n^0]\} \\ &= \prod_{n=1}^N [\rho_n^0\rho_n^1]^{1/2} \\ &= \prod_{n=1}^N b_n. \end{aligned}$$

Thus we have an *exact multiplicative decomposition* of the Törnqvist price index P_T into a product of N price effects, $\prod_{n=1}^N b_n$, where each price effect is a price index which shows the effect of changing just the nth price from p_n^0 to p_n^1.

We turn now to our second special case of (76) and (77) where g and h are defined by (28) for $r \neq 0$ and the restrictions (27) are satisfied. Thus f(q) is the quadratic mean of order r aggregator function defined by (29) for $r \neq 0$. Using (76) and (77) above, the generalized quadratic identity (34) in this case becomes:

(87)
$$\begin{aligned} [f^1]^r - [f^0]^r &= \sum_{n=1}^N [w_n^0\{f^0\}^r(q_n^0)^{1-r/2} + w_n^1\{f^1\}^r(q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}] \\ &= \sum_{n=1}^N \delta_n \end{aligned}$$

where δ_n defined in general by (77) becomes the following expression when the restrictions (27) and (28) are satisfied:

[25] See Diewert and Morrison (1986; 666-667) and Kohli (1990) for similar decompositions.

$$(88)\quad \delta_n = (1/2)\{[f(q_1^0,\ldots,q_{n-1}^0,q_n^1,q_{n+1}^0\ldots,q_N^0)]^r - [f(q_1^0,q_2^0,\ldots,q_N^0)]^r\}$$
$$+ (1/2)\{[f(q_1^1,q_2^1,\ldots,q_N^1)]^r - [f(q_1^1,\ldots q_{n-1}^1,q_n^0,q_{n+1}^1\ldots,q_N^1)]^r\}\ ; \quad n = 1,2,\ldots,N$$
$$= (1/2)[f^{\,0}]^r\{[\sigma_n^0]^r - 1\} + (1/2)[f^{\,1}]^r\{1 - [1/\sigma_n^1]^r\}$$

where the *quantity effects* σ_n^0 and σ_n^1 are defined by (79) and (80).

Now specialize (87) to the case where r = 1. Upon dividing both sides of (87) by $f^{\,0}$, we obtain the following additive percentage change decomposition for the implicit Walsh quantity index Q_1 defined earlier by (63):

$$(89)\quad Q_1 - 1 = \sum_{n=1}^N \delta_n / f^{\,0}$$

where $\delta_n / f^{\,0}$ is defined as

$$(90)\quad \delta_n / f^{\,0} \equiv (1/2)\{\sigma_n^0 - 1\} + (1/2)Q_1\{1 - [1/\sigma_n^1]\}\ ; \qquad n = 1,2,\ldots,N$$
$$= Q_{1n}\,[q_n^1 - q_n^0]$$

and where the *nth Walsh percentage change quantity weight* Q_{1n} was defined by (66). Thus the nth term in the additive percentage change decomposition for Q_1 given by (65), $Q_{1n}[q_n^1 - q_n^0]$, can be interpreted as a weighted sum of the percentage changes in the two single variable changes, $\sigma_n^0 - 1$ and $1 - [1/\sigma_n^1]$, where σ_n^0 and σ_n^1 are defined by (79) and (80). The weighted sum is an arithmetic average of the changes $\sigma_n^0 - 1$ and $1 - [1/\sigma_n^1]$ if the index Q_1 is equal to one.[26]

The above algebra can be adapted to provide an economic interpretation for the terms in the additive percentage change decomposition (69) that we obtained earlier for the Walsh implicit price index P_1. Thus we have

$$(91)\quad P_{1n}\,[p_n^1 - p_n^0] = (1/2)\{\rho_n^0 - 1\} + (1/2)P_1\{1 - [1/\rho_n^1]\}\ ; \qquad n = 1,2,\ldots,N$$

where the *nth Walsh percentage change price weight* P_{1n} was defined earlier by (70) and the *base period nth price effects* ρ_n^0 and the *current period nth price effects* ρ_n^1 were defined by (83) and (84). Thus the nth term in the additive percentage change decomposition for P_1 given by (69), $P_{1n}[p_n^1 - p_n^0]$, can be interpreted as a weighted sum of the percentage changes in the two single variable changes, $\rho_n^0 - 1$ and $1 - [1/\rho_n^1]$, where ρ_n^0 and ρ_n^1 are defined by (83) and (84). The weighted sum is an arithmetic average of the changes $\rho_n^0 - 1$ and $1 - [1/\rho_n^1]$ if the overall price index P_1 is equal to one.

Now specialize (87) to the case where r = 2. Upon dividing both sides of (87) by $f^{\,0}[f^{\,0} + f^{\,1}]$, we obtain the following additive percentage change decomposition for the Fisher ideal quantity index $Q_2 = Q_F$ defined earlier by (42):

[26] If σ_n^1 is close to one, then $1 - [1/\sigma_n^1]$ will be close to $\sigma_n^1 - 1$. These two expressions have the same first order Taylor series approximations around the point of approximation $\sigma_n^1 = 1$.

(92) $Q_F - 1 = \sum_{n=1}^{N} \delta_n / f^0[f^0 + f^1]$

where $\delta_n / f^0[f^0 + f^1]$ is defined for n = 1,2,…,N as

$$\begin{aligned}(93)\quad \delta_n / f^0[f^0 + f^1] &\equiv [(1/2)[f^0]^2\{[\sigma_n^0]^2 - 1\} + (1/2)[f^1]^2\{1 - [1/\sigma_n^1]^2\}/f^0[f^0 + f^1] \\ &= (1/2)\{f^0/[f^0 + f^1]\}\{[\sigma_n^0]^2 - 1\} + (1/2)Q_F\{f^1/[f^0 + f^1]\}\{1 - [1/\sigma_n^1]^2\} \\ &= (1/2)\{1/[1 + Q_F]\}\{[\sigma_n^0]^2 - 1\} + (1/2)[Q_F]^2\{1/[1 + Q_F]\}\{1 - [1/\sigma_n^1]^2\} \\ &= Q_{Fn}\,[q_n^1 - q_n^0]\end{aligned}$$

and where the *nth Fisher percentage change quantity weight* Q_{Fn} was defined by (47). Thus the nth term in the additive percentage change decomposition for Q_F given by (46), $Q_{Fn}[q_n^1 - q_n^0]$, can be interpreted as a weighted sum of the changes in the two single variable changes, $[\sigma_n^0]^2 - 1$ and $1 - [1/\sigma_n^1]^2$, where σ_n^0 and σ_n^1 are defined by (79) and (80). If the Fisher quantity index Q_F equals 1, then (93) becomes:

$$\begin{aligned}(94)\quad Q_{Fn}\,[q_n^1 - q_n^0] &= (1/4)\{[\sigma_n^0]^2 - 1\} + (1/4)\{1 - [1/\sigma_n^1]^2\} \\ &= (1/4)\{[\sigma_n^0 - 1][\sigma_n^0 + 1]\} + (1/4)\{[1 - (1/\sigma_n^1)][1 + (1/\sigma_n^1)]\} \\ &\approx (1/2)[\sigma_n^0 - 1] + (1/2)[1 - (1/\sigma_n^1)]\end{aligned}$$

where the last approximation follows if the two quantity effects σ_n^0 and σ_n^1 are close to one. Thus under normal conditions when all of the quantity indexes are close to one, the nth term in the additive percentage change decomposition for Q_F given by (46), $Q_{Fn}[q_n^1 - q_n^0]$, will be approximately equal to the arithmetic average of the two single variable index changes, $\sigma_n^0 - 1$ and $1 - (1/\sigma_n^1)$.[27]

Of course, the above algebra can be adapted to provide an economic interpretation for the terms in the additive percentage change decomposition (48) that we obtained earlier for the Fisher price index $P_2 = P_F$. Thus we have for n = 1,2,…,N:

$$(95)\quad P_{Fn}\,[p_n^1 - p_n^0] = (1/2)\{1/[1+P_F]\}\{[\rho_n^0]^2 - 1\} + (1/2)[P_F]^2\{1/[1+P_F]\}\{1 - [1/\rho_n^1]^2\}$$

where the *nth Fisher percentage change price weight* P_{Fn} was defined earlier by (50) and the *base period nth price effects* ρ_n^0 and the *current period nth price effects* ρ_n^1 were defined by (83) and (84). Thus if P_F, ρ_n^0 and ρ_n^1 are all close to one, then the nth term in the additive percentage change decomposition for P_F given by (48), $P_{Fn}[p_n^1 - p_n^0]$, is approximately equal to the arithmetic average of the percentage changes in the two single variable changes, $\rho_n^0 - 1$ and $1 - [1/\rho_n^1]$, where ρ_n^0 and ρ_n^1 are defined by (83) and (84).

8. Conclusion

The results in the previous sections demonstrate that the quadratic identity (4) and its generalizations provide a unifying framework for deriving all of the most commonly used

---

[27] Thus under these conditions, the terms in (90) will closely approximate the terms in (93).

superlative index number formula. In addition, the single variable quadratic identity (73) and its generalizations have proven to be very useful in providing economic interpretations for some additive percentage change decompositions for these commonly used superlative indexes.[28]

References

Christensen, L.R., D.W. Jorgenson and L.J. Lau (1971), "Conjugate Duality and the Transcendental Logarithmic Production Function," *Econometrica* 39, 255-256.

Christensen, L.R., D.W. Jorgenson and L.J. Lau (1973), "Transcendental Logarithmic Production Frontiers", *Review of Economics and Statistics* 55, 28-45.

Diewert, W.E., 1974. "Applications of Duality Theory," pp. 106-171 in M.D. Intriligator and D.A. Kendrick (ed.), Frontiers of Quantitative Economics, Vol. II, Amsterdam: North-Holland.

Diewert, W.E. (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1993), "Duality Approaches To Microeconomic Theory", in *Essays in Index Number Theory*, pp. 105-175 in Volume I, Contributions to Economic Analysis 217, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North Holland.

Diewert, W.E. (2000), "The Consumer Price Index and Index Number Purpose", Discussion Paper 00-02, Department of Economics, University of British Columbia, Vancouver, Canada, V6T 1Z1, 69 pp.

Diewert, W.E. and C.J. Morrison (1986), "Adjusting Output and Productivity Indexes for Changes in the Terms of Trade", *The Economic Journal* 96, 659-679.

Dikhanov, Y. (1997), "The Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures", mimeo, International Economics Department, The World Bank, Washington D.C., January.

Ehemann, C., A.J. Katz and B.R. Moulton (2000), "The Chain-Additivity Issue and the U.S. National Accounts", mimeo, Bureau of Economic Analysis, U.S. Department of Commerce, Washington D.C. 20230, September.

Färe, R. and K.J. Sung (1986), "On Second Order Taylor's Series Approximation and Linear Homogeneity", *Aequationes Mathematicae* 30, 180-186.

Fisher, I. (1911), *The Purchasing Power of Money,* London: Macmillan.

[28] For the Törnqvist price and quantity indexes, we obtained multiplicative decompositions rather than additive ones.

Fisher, I. (1922), *The Making of Index Numbers*, Houghton-Mifflin, Boston.

Kohli, U. (1990), "Growth Accounting in the Open Economy: Parametric and Nonparametric Estimates", *Journal of Economic and Social Measurement* 16, 125-136.

Konüs, A.A. (1924), "The Problem of the True Index of the Cost of Living", translated in *Econometrica* 7, (1939), 10-29.

Knibbs, Sir G.H. (1924), "The Nature of an Unequivocal Price Index and Quantity Index", *Journal of the American Statistical Association* 19, 42-60 and 196-205.

Lau, L.J. (1979), "On Exact Index Numbers", *Review of Economics and Statistics* 61, 73-82.

Moulton, B.R., and E.P. Seskin (1999), "A Preview of the 1999 Comprehensive Revision of the National Income and Product Accounts", *Survey of Current Business* 79 (October), 6-17.

Reinsdorf, M.B., W.E. Diewert and C. Ehemann (2000), "Additive Decompositions for the Fisher, Törnqvist and Geometric Mean Indexes", unpublished paper.

Samuelson, P.A. and S. Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64, 566-593.

Shephard, R.W. (1953), *Cost and Production Functions*, Princeton: Princeton University Press.

Törnqvist, Leo (1936), "The Bank of Finland's Consumption Price Index," *Bank of Finland Monthly Bulletin* 10: 1-8.

Törnqvist, L. and E. Törnqvist (1937), "Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

Van Ijzeren (1987), *Bias in International Index Numbers: A Mathematical Elucidation*, Den Haag: Koninklijke Bibliotheek.

Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.

Walsh, C.M. (1921), *The Problem of Estimation*, London: P.S. King & Son.

Wold, H. (1943) (1944), “A Synthesis of Pure Demand Analysis”, *Skandinavisk Altuarietidskrift* 26, 85-144 and 220-275 and 27, 69-120.

Wold, H. (1953), *Demand Analysis*, New York: John Wiley.